Requested Solutions (§ 14.8)

(1c) Use Lagrange Multipliers to optimize

$f(x,y,z) = \ln(x^2+1) + \ln(y^2+1) + \ln(z^2+1)$ subject to $x^2+y^2+z^2 = 12$.

Sol: Consider function $H(x,y,z,\lambda) = \ln(x^2+1) + \ln(y^2+1) + \ln(z^2+1) + \lambda(x^2+y^2+z^2-12)$.

$$\nabla H = \left\langle \frac{2x}{x^2+1} + 2x\lambda,\ \frac{2y}{y^2+1} + 2y\lambda,\ \frac{2z}{z^2+1} + 2z\lambda,\ x^2+y^2+z^2-12 \right\rangle$$

As solutions must satisfy $\nabla H = \vec{0}$ we have

$$\nabla H = \vec{0} \iff \begin{cases} 2x\left(\frac{1}{x^2+1} + \lambda\right) = 0 \\ 2y\left(\frac{1}{y^2+1} + \lambda\right) = 0 \\ 2z\left(\frac{1}{z^2+1} + \lambda\right) = 0 \\ x^2+y^2+z^2 = 12 \end{cases} \iff \begin{cases} x = 0 \quad \text{OR} \quad \lambda = -\frac{1}{x^2+1} \\ y = 0 \quad \text{OR} \quad \lambda = -\frac{1}{y^2+1} \\ z = 0 \quad \text{OR} \quad \lambda = -\frac{1}{z^2+1} \\ x^2+y^2+z^2 = 12 \end{cases}$$

Now we need to make a case analysis depending on which of x,y,z is 0.

Note: $f(x,y,z)$ is symmetric in x, y, and z (i.e. the order of inputs doesn't affect output). Thus we can analyze cases by number of zeroes as follows

Case 1: $x=y=z=0$ implies $0 = x^2+y^2+z^2 = 12$, so reject this case.

Case 2: $x=y=0$ implies $z^2 = 12$, so we have $f(x,y,z) = \ln(13)$ in this case.

Case 3: $x=0$, $y \neq 0 \neq z$ implies $-\frac{1}{y^2+1} = \lambda = -\frac{1}{z^2+1}$ and $y^2+z^2 = 12$.

Thus $-(y^2+1) = -(z^2+1)$ yields $y^2 = z^2 = 6$. Hence $f(x,y,z) = 2\ln(7) = \ln(49)$ in this case.

Case 4: $x,y,z \neq 0$ implies $\lambda = -\frac{1}{x^2+1} = -\frac{1}{y^2+1} = -\frac{1}{z^2+1}$. Thus $x^2 = y^2 = z^2$, so

$x^2+y^2+z^2 = 12$ yields $x^2 = y^2 = z^2 = 4$. Hence $f(x,y,z) = 3\ln(5) = \ln(125)$ in this case.

Thus the values we must consider are $\ln(13) < \ln(49) < \ln(125)$.

Hence the absolute minimum is $\ln(13)$, and the absolute maximum is $3\ln(5)$. ∎

NB: You could also rewrite $f(x,y,z) = \ln\left((x^2+1)(y^2+1)(z^2+1)\right)$, note that $\ln$ is a strictly increasing function, and optimize $(x^2+1)(y^2+1)(z^2+1)$ subject to $x^2+y^2+z^2 = 12$ using Lagrange multipliers.

(2b) Find extreme values of $f(x,y,z) = xy + yz$ subject to $xy = 1$ and $y^2+z^2=1$.

Sol: Consider the function $H(x,y,z,\lambda,\mu) = xy + yz + \lambda(1-xy) + \mu(1-y^2-z^2)$.

Extreme values occur when $\nabla H = \vec{0}$.

$$\nabla H = \vec{0} \iff \begin{cases} y - y\lambda = 0 \\ x + z - \lambda x - 2\mu y = 0 \\ y - 2\mu z = 0 \\ 1 - xy = 0 \\ 1 - y^2 - z^2 = 0 \end{cases} \iff \begin{cases} y(1-\lambda) = 0 & (1) \\ x - \lambda x - 2\mu y + z = 0 & (2) \\ y - 2\mu z = 0 & (3) \\ xy = 1 & (4) \\ y^2 + z^2 = 1 & (5) \end{cases}$$

Note (4) $\Rightarrow y \neq 0$, so $\lambda = 1$ by (1). Thus our system reduces to

$$\nabla H = \vec{0} \iff \begin{cases} \lambda = 1 & (1^*) \\ z = 2\mu y & (2^*) \\ y = 2\mu z & (3) \\ x = \frac{1}{y} & (4) \\ y^2+z^2 = 1 & (5) \end{cases}$$

Now (2^*) and (3) yield

$$y = 2\mu z = 2\mu \cdot 2\mu y$$

$\therefore \mu^2 = \frac{1}{4}$, so $\mu = \pm \frac{1}{2}$.

Case 1: If $\mu = \frac{1}{2}$ we obtain by (5):

Signs match by (2^*) and (4).

$$1 = y^2 + z^2 = y^2 + \left(2 \cdot \tfrac{1}{2} y\right)^2 = 2y^2 \implies y = \pm \frac{1}{\sqrt{2}} \Rightarrow z = \pm \frac{1}{\sqrt{2}},\ x = \pm\sqrt{2}.$$

Hence we obtain two test points $\left(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Case 2: If $\mu = -\frac{1}{2}$ we obtain by (5)

Signs differ by (2^*)

$$1 = y^2 + z^2 = y^2 + \left(2 \cdot -\tfrac{1}{2} y\right)^2 = 2y^2 \implies y = \pm \frac{1}{\sqrt{2}} \implies x = \pm\sqrt{2},\ z = \mp \frac{1}{\sqrt{2}}$$

signs match by (4)

Hence we obtain two more test points $\left(\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, $\left(-\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Finally we evaluate test points to obtain

P	$\left(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\left(\sqrt{2}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$\left(-\sqrt{2}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\left(-\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
$f(P)$	$\frac{1}{\sqrt{2}}\left(\sqrt{2} + \frac{1}{\sqrt{2}}\right) = \frac{3}{2}$	$\frac{1}{\sqrt{2}}\left(\sqrt{2} - \frac{1}{\sqrt{2}}\right) = \frac{1}{2}$	$-\frac{1}{\sqrt{2}}\left(-\sqrt{2} + \frac{1}{\sqrt{2}}\right) = \frac{1}{2}$	$-\frac{1}{\sqrt{2}}\left(-\sqrt{2} - \frac{1}{\sqrt{2}}\right) = \frac{3}{2}$

$\therefore \frac{3}{2}$ is the absolute max value and $\frac{1}{2}$ is the absolute min value. ///

③ Find the points on the intersection of $x+y+2z=2$ and $z=x^2+y^2$ which are nearest to and furthest from the origin.

Sol: Optimizing distance is equivalent to optimizing square distance (because the square root is a strictly increasing function).

Thus we seek to optimize $x^2+y^2+z^2$ subject to $x+y+2z=2$ and $z=x^2+y^2$.

Consider $H(x,y,z,\lambda,\mu) = x^2+y^2+z^2 + \lambda(x+y+2z-2) + \mu(x^2+y^2-z)$.

$$\nabla H = \vec{0} \iff \begin{cases} 2x + \lambda + 2\mu = 0 \\ 2y + \lambda + 2\mu = 0 \\ 2z + 2\lambda - \mu = 0 \\ x + y + 2z = 2 \\ x^2 + y^2 = z \end{cases} \iff \begin{cases} x = -\frac{1}{2}(\lambda + 2\mu) = y \\ z = \frac{1}{2}(\mu - 2\lambda) \\ 2x + 2z = 2 \\ z = 2x^2 \end{cases}$$

$$\iff \begin{cases} x = y = -\frac{1}{2}(\lambda + 2\mu) & (1) \\ z = \frac{1}{2}(\mu - 2\lambda) & (2) \\ x + z = 1 & (3) \\ z = 2x^2 & (4) \end{cases}$$

Using (3) and (4) we obtain $1-x = z = 2x^2$, so $2x^2 + x - 1 = 0$.

Hence $x = \dfrac{-1 \pm \sqrt{1^2 - 4(2)\cdot(-1)}}{2\cdot 2} = \dfrac{-1 \pm \sqrt{9}}{4} = \dfrac{-1 \pm 3}{4}$, yielding $x = \frac{1}{2}$ OR $x = -1$.

Hence we obtain two points to test (from $y = x$, $z = 1-x$):

$$\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) \text{ and } (-1,-1,2).$$

$$\begin{cases} d\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} \\ d(-1,-1,2) = (-1)^2 + (-1)^2 + 2^2 = 6 \end{cases}$$

Hence $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the point closest to the origin and $(-1,-1,2)$ is the point furthest from the origin on the curve of intersection of $x+y+2z=2$ and $z=x^2+y^2$.